Elaborate Solutions for Some Questions in Exercise Sheet 6

January 4, 2013

1. Let $z \in U$ be an arbitrary point. Since U is open there is $\epsilon > 0$ such that $D_\epsilon(z) \subset U$. For all $n \in \mathbb{N}$, f_n is holomorphic in $D_\epsilon(z)$. Let γ be an arbitrary closed path in $D_\epsilon(z)$. For all $n \in \mathbb{N}$, the integral of f_n around γ is zero because $D_\epsilon(z)$ is simply connected. By Weierstrass M-test $\sum_{n=1}^{\infty} f_n$ is uniformly convergent. Now it is possible to switch integral and sum to show that the integral around γ of the function $\sum_{n=1}^{\infty} f_n$ is zero. Also notice that $\sum_{n=1}^{\infty} f_n$ is continuous as the uniformly convergent sum of continuous functions. By Morera's Theorem $\sum_{n=1}^{\infty} f_n$ is holomorphic in $D_\epsilon(z)$. Since z was arbitrary we get that $\sum_{n=1}^{\infty} f_n$ is holomorphic in U.

2. Using Fubini's theorem to switch the order of integration and show that integrals around closed curves in $Re(z) > 0$ of the function Γ are zero. Also notice that Γ is continuous. By Morera's Theorem it is holomorphic in $Re(z) > 0$.

3. Since f is non constant in U. There is a point $z_0 \in U$ such that $f'(z_0) \neq 0$. $(\exp(f(z_0)))' = \exp(f(z_0)) \cdot f'(z_0) \neq 0$ because the exp is never zero. Therefore $\exp(f)$ is non constant in U, it is also holomorphic in U. By the Maximum Modulus $\exp(f)$ has no local maximum in U. Observe that local maximum of the real part of f is also a local maximum of $\exp(f)$. Therefore the real part of f does not have a local maximum in U.

4. Assume in contradiction that $f(z_0) \neq 0$. Since z_0 is a local minimum there exist an open neighborhood $V \subset U$ of z_0 such that $|f(z_0)| \leq |f(z)|$ for all $z \in V$. f is never zero in V. Therefore $\frac{1}{f}$ is holomorphic in V. z_0 is a local maximum of $\frac{1}{f}$. By the Maximum Modulus $\frac{1}{f}$ is contant in U. Thefore f is constant in U, a contradiction.